

EXPONENTIALS AND R -RECURRENT RANDOM WALKS ON GROUPS

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ABSTRACT. On a locally compact group E with countable base, we consider a random walk X that has a unique (up to a positive factor) r -invariant measure for some $r > 0$. Under some weak conditions on the measure, there is a unique continuous exponential on E naturally associated with X . It follows that there exists an R -recurrent random walk in the sense of Tweedie on E if and only if E is a recurrent group and there exists a Harris random walk on E .

1. INTRODUCTION

We deal with random walks on locally compact groups with countable base. All groups considered below are assumed, without exception, to have these properties, and the group operation is always written as multiplication. The terminology pertaining to random walks and irreducible Markov chains is mainly borrowed from the books [8] and [7], respectively.

Now let us start the exposition. It is well known that substantial attention has recently been paid to the description of recurrent groups, that is, groups on which there exists at least one recurrent random walk. Most progress in this direction has been made for abelian groups and connected Lie groups (see [3], [8, Chap. 3], and [2]). In particular, it has turned out that recurrent groups possess some special properties. (In particular, they are unimodular [4].)

In the present paper, we single out a fairly large family of random walks that are not necessarily recurrent but can only be realized on recurrent groups (see Theorem 2 below). This family consists of all possible spread out random walks that are R -recurrent in the sense of Tweedie, or, which is the same, are $\tilde{\pi}$ -irreducible R -recurrent Markov chains [7, 10], where $\tilde{\pi}$ is the Haar measure on the group and $R \geq 1$ is the convergence parameter of the Markov chain. Needless to say, such a random walk is ρ -recurrent in the sense common in random walk theory. (See [11, Chap 2]; here ρ is the spectral radius of the random walk.) An arbitrary Bernoulli random walk on the integer lattice generated by Bernoulli trials with success probability $\rho \neq 0, 1$ can serve as an example of an R -recurrent random walk.

The above-mentioned Theorem 2 is preceded by Theorem 1 concerning conditions for the existence of a unique continuous exponential associated with some random walk (see (1) below). In this connection, recall that a Borel function $\varphi > 0$ defined on a group E is called an *exponential* on E if $\varphi(xy) = \varphi(x)\varphi(y)$ for any

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$x, y \in E$; such functions play a noticeable role in random walk theory [11]. Theorem 2 expresses the simple fact that if a random walk is spread out and R -recurrent in the sense of Tweedie, then the random walk on the same group corresponding to the law $R\varphi v$, where v is the law of the original random walk, is a Harris random walk in the standard sense [8, 6].

Theorem 1 and 2, which are proved in Secs. 2 and 4, respectively, are the main results of the present paper. The other assertions, which are mainly gathered in Sec. 3, supplement Sec. 1 and contain preliminary material for Sec. 4.

In a subsequent publication, the author intends to use the theory discussed here to further develop the results in [9], in particular, to obtain new strong ratio limit theorems.

Let us explain the main notation used in the paper. We everywhere consider a group E of the type indicated above with the family \mathcal{E} of Borel subsets. We fix a right Haar measure π on \mathcal{E} and the corresponding left Haar measure π_1 such that $\pi_1(A) = \pi(A^{-1})$ for any $A \in \mathcal{E}$, where $A^{-1} = \{x \in E: x^{-1} \in A\}$. The abbreviation “a.e.” stands for “ π -almost everywhere” or “ π -almost every,” depending on the context.

We specify a random walk $X = (X_n; n \geq 0)$ on E , which will be subjected to various restrictions where necessary. We assign the random walk \hat{X} dual to X to the law \hat{v} , where $\hat{v}(A) = v(A^{-1})$, $A \in \mathcal{E}$, and the transition operators corresponding to X and \hat{X} are denoted by P and \hat{P} , respectively [8]. Thus,

$$Pf(x) = \int f(xy)v(dy), \quad \hat{P}f(x) = \int f(xy)\hat{v}(dy)$$

for all $x \in E$, where f ranges over the family of all Borel functions $f: E \rightarrow (-\infty, \infty]$ bounded below.

Finally, if ν is a measure on \mathcal{E} and f is a nonnegative Borel function on E , then the measures νP and $\mu = f\nu$ are defined in the usual way,

$$\nu P(A) = \int p(x, A)\nu(dx), \quad \mu(A) = \int_A f d\nu, \quad A \in \mathcal{E}.$$

2. EXPONENTIALS AND RANDOM WALKS

Let $r \in (0, \infty)$. A Borel function $f: E \rightarrow [0, \infty]$ or a measure ν defined on \mathcal{E} is said to be *r -invariant for a random walk X* if $\int f d\pi > 0$, $f \not\equiv \infty$, and $f = rPf$ or if $\nu(E) > 0$ and $\nu = r\nu P$, respectively [7]. (Many authors give a different interpretation to similar notions; e.g., cf. the definition of an invariant (or, which is the same, harmonic) function in [1, 7, 5].)

An exponential φ defined on E is an r -invariant function for X if and only if

$$(1) \quad r = \left[\int \varphi dv \right]^{-1},$$

because

$$P\varphi(x) = \int \varphi(x)\varphi(y)v(dy) = \varphi(x) \int \varphi dv.$$

The same assertion holds for the case of \hat{X} except that Eq. (1) should be replaced with its counterpart that contains the measure \hat{v} instead of v and is naturally called the dual version of (1).

Theorem 1. *Assume that, for some $r > 0$, a random walk X has a unique (up to a positive factor) r -invariant measure π_0 that is continuous with respect to π and takes finite values on compact subsets of E . Then there exists a unique continuous exponential φ on E satisfying condition (1), and the function $\psi = \varphi^{-1}$ is the unique continuous exponential on E satisfying the dual version of (1). The exponentials φ and ψ are r -invariant for X and \hat{X} , respectively, and the measure π_0 coincides with $\psi\pi$ up to a positive factor.*

Proof. By the assumption of the theorem, $\pi_0 = h_1\pi$ for some Borel function locally π -integrable (i.e., π -integrable on any compact subset of E). Let $v \geq 0$ ($v \not\equiv 0$) be a continuous compactly supported function on E . Then the function

$$h(x) = \int v(y)h_1(yx)\pi(dy) = \int v(yx^{-1})h_1(y)\pi(dy), \quad x \in E,$$

takes finite values and cannot be zero on the entire E . Moreover, it is continuous on E by virtue of the last relation.

Let us verify that

$$(2) \quad \int f d\mu = r \int Pf d\mu$$

for the measure $\mu = h\pi$ and for every Borel function $f \geq 0$. Indeed, if Δ is the modular function of E , then

$$\begin{aligned} \int f(x) d\mu &= \Delta(y) \int f(y^{-1}x)h_1(x)\pi(dx) = \Delta(y) \int f(y^{-1}x)\pi_0(dx) \\ &= r\Delta(y) \int f(y^{-1}x)\pi_0 P(dx) = r\Delta(y) \int Pf(y^{-1}x)\pi_0(dx) \\ &= r \int Pf(x)h_1(yx)\pi(dx) \end{aligned}$$

for every $y \in E$ by [5, Theorem 15.15], where the third equality takes into account the r -invariance of π_0 and the last equality uses the first three with f replaced by Pf . Moreover,

$$\begin{aligned} \int f d\mu &= \int f(x) \left[\int v(y)h_1(yx)\pi(dy) \right] \pi(dx) \\ (3) \quad &= \int v(y) \left[\int f(x)h_1(yx)\pi(dx) \right] \pi(dy) \\ &= r \int v(y) \left[\int Pf(x)h_1(yx)\pi(dx) \right] \pi(dy) \end{aligned}$$

according to the preceding computation. The first two equalities in (3) with Pf substituted for f show that the right-hand side of (2) is equal to the last expression in (3). Hence (3) implies relation (2), which shows that the measure μ is R -invariant.

Let us verify that

$$(4) \quad h(yx) = a(x)h(y)$$

for any $x, y \in E$ but for now postpone the determination of $a(y) > 0$. To this end, we take a point $y \in E$ and a function f of the same type as above and write

out (2) with $f(x)$ replaced by $f(y^{-1}x)$. As a result, we readily find that

$$\int f(x)h(yx)\pi(dx) = r \int Pf(x)h(yx)\pi(dx)$$

for a broad class of functions f , and hence the measure μ_y , where $\mu_y(dx) = h(yx)\pi(dx)$, is r -invariant for X . Consequently, μ_y coincides with π_0 and μ up to some factors, so that $\mu_y = a(y)\mu$ for some $a(y) > 0$. In other words, for each $y \in E$, Eq. (4) holds for a.e. $X \in E$, and since h is continuous, it follows that Eq. (4) holds for any $x, y \in E$.

For $x = e$, where e is the identity element of E , it follows from (4) that $h(y) = a(y)h(e)$, $y \in E$, and so $h(e) \neq 0$ and $a(y) = h(y)/h(e)$. Accordingly, replacing the function v used in the definition of h by the function αv with some $\alpha > 0$ if necessary, we can assume in what follows that $h(e) = 1$ and $a(y) = h(y)$ on E , and Eq. (4) means in this case that h is a continuous exponential on E .

The function $\varphi = h^{-1}$ is a continuous exponential as well, and moreover, it satisfies condition (1). Indeed, by using the random walk \widehat{X} dual to X , we can rewrite (2) in the form

$$(5) \quad \int fh \, d\pi = r \int f\widehat{P}h \, d\pi,$$

where f ranges over the same family of functions as above. We see that $h = r\widehat{P}h$ and hence

$$(6) \quad \begin{aligned} h(x) &= r\widehat{P}h(x) = r \int h(xy)v(dy) = rh(x) \int \frac{\widehat{v}(dy)}{h(y^{-1})} \\ &= rh(x) \int \frac{\widehat{v}(dy)}{h(y^{-1})} = rh(x) \int \frac{v(dy)}{h(y)} = rh(x) \int \varphi \, dv \end{aligned}$$

for a.e. $x \in E$, because h is an exponential and $h(y^{-1}) = h^{-1}(y) = \varphi(y)$. By comparing the left- and rightmost expressions in (6), we arrive at condition (1) and hence to the r -invariance of φ for X .

We point out that all but the first equality in (6) is a priori satisfied everywhere in E , and hence all terms of these equalities, together with the last term, are continuous on E . Since h is continuous, it follows that all equalities in (6) hold everywhere in E . The first of them establishes the r -invariance of the exponential $h = \varphi^{-1} = \psi$ for \widehat{X} and hence the validity of the dual version of (1) for ψ .

Next, if some continuous exponential ψ_1 satisfies the same version of (1), then $\psi_1 = r\widehat{P}\psi_1$, whence one again obtains (5) with ψ_1 substituted for h . In other words, we can apply (2) with μ replaced by $\mu_1 = \psi_1\pi$, thus establishing the r -invariance of μ_1 for X together with the relation $\varphi_1 = k\psi$ for an appropriate $k > 0$. However, $\psi_1(e) = \psi(e) = 1$; i.e., $k = 1$ and $\psi_1 = \psi$. Thus, ψ has the uniqueness property claimed in the theorem.

One can readily justify a similar property of the exponential ϕ . Namely, if a continuous exponential ϕ_1 satisfies condition (1), then the dual version of (1) applies to $\psi_2 = \phi_1^{-1}$, because

$$\int \psi_2 \, d\widehat{v} = \int \psi_2(x^{-1})v(dx) = \int \psi_1 \, dv = r^{-1},$$

and by the preceding we have $\psi_2 = \psi$ and $\varphi_1 = \psi^{-1} = \varphi$. The proof is complete. \square

Corollary 1. *If, under the assumptions of Theorem 1, some function g is r -invariant for \widehat{X} and locally π -integrable, then $g = k\varphi$ a.e. for some $k > 0$. If the assumptions of the theorem hold for the random walk \widehat{X} , then every function g that is r -invariant for X and locally π -integrable has the form $g = k_1\varphi$ a.e. for some $k_1 > 0$.*

To establish the first part of the corollary, it suffices to notice that the measure $g\pi$ is in this case r -invariant for X and takes finite values on compact sets; hence it coincides, up to some factors, with the measures π_0 and $\mu = \varphi^{-1}\pi$ in the proof of the theorem. The second part of the corollary follows from the first.

There is a more convenient version of this corollary for spread out R -recurrent random walks (see Proposition 5). As to the condition that r -invariant functions be locally integrable, it is often satisfied automatically (see Proposition 2).

Corollary 2. *Let the random walk X be symmetric (i.e., $v = \widehat{v}$), and assume that, for some $r > 0$, any of its r -invariant measures can differ only by appropriate factors. Then $r = 1$, and every invariant (i.e., r -invariant) function for X is a.e. equal to some constant.*

Indeed, in this case we can assume that $X = \widehat{X}$, and Theorem 1 implies the identity $\varphi \equiv \varphi^{-1}$, by which $\varphi = 1$ and $r = 1$ (see (1)). The remaining part of Corollary 2 can be derived from Corollary 1.

3. INTERMEDIATE RESULTS

In this section, on the one hand, we make some preparations for the proof of Theorem 2 in Sec. 4; on the other hand, we give supplementary material related to Theorem 1 and Corollary 1 (see Proposition 2).

In all subsequent statements except for Proposition 4, X is assumed to be a spread out random walk, that is, a random walk for which some convolution power v^n , $n \geq 1$, of the law v is nonsingular with respect to the Haar measure π . Moreover, as is often done (cf. [8]), we restrict ourselves to adapted random walks.

Let us start by discussing the irreducibility property, which is interpreted differently in random walk theory and in the general Markov chain theory. Namely, a random walk is said to be *irreducible* [1, 8] if the least closed semigroup $T \subset E$ containing the support of the measure v coincides with E ; it is said to be π -irreducible [7] if

$$(7) \quad \sum_{n \geq 1} p(n, x, A) > 0$$

for all $x \in E$ and $A \in \mathcal{E}_+$, where $p(n, \cdot, \cdot)$ are the transition probabilities in n steps corresponding to X and $\mathcal{E}_+ = \{A \in \mathcal{E} : \pi(A) > 0\}$.

The proof of Proposition 1, which establishes that these two definitions are essentially the same in the case of spread out random walks, and of Propositions 2 and 3 is based on Lemma 3.7 in [8, Chap. 3], which establishes the existence of a compact set $V \subset E$ with nonempty interior, a positive integer m , and positive

numbers a and b such that

$$(8) \quad p(m, x, A) = v^m(x^{-1}A) \geq a\pi(V \cap x^{-1}A) \geq b\pi_1(V \cap x^{-1}A)$$

for any $x \in E$ and $A \in \mathcal{E}$. (See the end of Sec. 1 for the definition of a left Haar measure.)

Proposition 1. *A spread out random walk X is π -irreducible if and only if it is irreducible.*

Proof. Assume that X is irreducible and use relations (8) for $x \in E$, $A \in \mathcal{E}_+$, and V, m , etc. chosen according to what was just said. If $A \in \mathcal{E}_+$ is relatively compact, then the last expression in (8) continuously depends on $x \in E$ and does not vanish identically [5, Corollary 20.17], so that $p(m, x, A) > 0$ for all x in a nonempty open set. It follows from the Chapman–Kolmogorov equation and the irreducibility of X that for each $x \in E$ one has $p(n, x, A) > 0$ for some positive integer $n = n(x)$, and this conclusion readily extends to arbitrary $A \in \mathcal{E}_+$. Thus, we have established the π -irreducibility of X .

Conversely, let X be π -irreducible, and let T be the above-mentioned semigroup. The set $S = E \setminus T$ is open, and hence $\pi(S) > 0$ provided that S is nonempty. But then $\sum_{n \geq 1} p(n, e, S) > 0$ by (7), which contradicts the definition of T . Thus, S is empty; i.e., the random walk X is irreducible. \square

Corollary 3. *If a spread out random walk X is irreducible, then each of the Haar measures π and π_1 is a maximal irreducibility measure for X in the sense of [7].*

This corollary, which is a straightforward consequence of Proposition 1, permits freely using the theory developed in [7] in the subsequent exposition.

Note also that, according to Proposition 1, we can replace the condition of π -irreducibility in the definition of R -recurrent random walk (see Sec. 1) by the condition of irreducibility as long as we restrict ourselves to spread out random walks.

Recall that the replacement of the equality $f = rPf$ in the definition of r -invariant function (see Sec. 1) by the inequality $f \geq rPf$ gives the definition of r -subinvariant function [7].

Proposition 2. *If a random walk X is spread out and irreducible, then, for each $r > 0$, every function r -subinvariant for X is locally π -integrable.*

Proof. Assume the contrary: there exists an r -subinvariant function f for X that is not π -integrable on a nonempty compact set F . Then f is not π -integrable and hence not π_1 -integrable in any neighborhood of some point $z \in F$. (Otherwise, it would be π -integrable in some neighborhood G_x of each point $x \in F$, and finitely many sets of the form G_x , $x \in F$, would form an open cover of the compact set F , which is only possible if f is π -integrable on F .)

Fix a point z with this property and again use relation (8) retaining the preceding notation. Since the group E is regular [5, Chap. 1], it follows that there exists a nonempty open set W whose closure is contained in the set V indicated in (8) and a neighborhood W_0 of the identity element e such that $x^{-1}W \subset V$ for all $x \in W_0$. For any $x \in W_0$ and any Borel set $A \subset W$, the last expression in (8) coincides with

$$(9) \quad b\pi_1(x^{-1}A) = b\pi_1(A),$$

because π_1 is a left Haar measure.

Now take a $y \in W$, set $s = yz^{-1}$ and $g(x) = f(s^{-1}x)$, $x \in E$, and note that

$$\begin{aligned} \int_W g d\pi_1 &= \int 1_W(x) f(s^{-1}x) \pi_1(dx) \\ &= \int 1_W(sx) f(x) \pi_1(dx) = \int_{s^{-1}W} f d\pi_1 = \infty \end{aligned}$$

(where 1_W is the indicator function of the set W), because $s^{-1}W$ is a neighborhood of the point $s^{-1}y = z_0$ and f is nonintegrable in any neighborhood of that point. Hence we have, by (8) and (9),

$$P^m g(x) \geq b \int_W g d\pi_1 = \infty, \quad x \in W_0,$$

and consequently, $P^m f(x) = P^m g(sx) = \infty$ if x ranges over the open set $s^{-1}W_0$. Clearly, $f \equiv \infty$ on the same set by virtue of the r -subinvariance of f ; in conjunction with the Chapman–Kolmogorov equation, this implies that $f \equiv \infty$ on E (cf. the preceding proof), which is inconsistent with the definition of r -subinvariant function. Thus, our assumption at the beginning of the proof is wrong, and the proof is complete. \square

Proposition 3. *If a random walk X is spread out and irreducible, then there exists an open set $U \subset E$ such that the set sU and each of the measures $1_{sU}\pi_1$ and $1_{sU}\pi$ is small for X in the sense of [7].*

Proof. Relations (8) and (9) imply the inequality

$$(10) \quad p(m, x, A) \geq b\pi_1(A),$$

where x ranges over some neighborhood W_0 of e , the set A is an arbitrary Borel subset of a nonempty open set $W \subset E$, and a positive integer m and a $b > 0$ are chosen appropriately. Furthermore, without loss of generality we can assume that W_0 is relatively compact. According to [7, Definition 2.3], inequality (10) shows that the function 1_{W_0} and the measure $1_{W_0}\pi_1$ are small for X . Now take an $s \in E$. If $x \in sW_0$ and $A \subset sW$ ($A \in \mathcal{E}$), then, by (10),

$$p(m, x, A) = p(m, s^{-1}x, s^{-1}A) \geq b\pi_1(s^{-1}A) = b\pi_1(A),$$

because $s^{-1}x \in W_0$ and $s^{-1}A \subset W$. Thus, we again arrive at (10) but with somewhat different x and A , and this time the set sW_0 and the measure $1_{sW_0}\pi_1$ prove to be small. In view of the relative compactness of W and the relation between the right and left Haar measures [5, Sec. 15], we see that the measure $1_{sW}\pi$ is small as well. To complete the proof, it remains to set $U = W_0 \cap s_0W$ for an $s_0 \in E$ such that U is nonempty. \square

In the following section, we need an intuitively clear statement (see Proposition 4) which will help us establish that a certain random walk is a Harris random walk. The proof of this statement is based on the following lemma, where the symbol M_x , $x \in E$, stands for the expectation corresponding to the probability measure P_x (which is defined on the corresponding σ -algebra of events and is assigned to X for the initial state x [7, 8]).

Lemma 1. *For every bounded Borel function g on E , the relation*

$$(11) \quad M_{yx}[g(X_n)] = M_x[g(yX_n)], \quad n \geq 0,$$

holds for any $x, y \in E$.

Proof. For $n = 0$, this is obvious, so consider the case of $n \geq 1$. If $g = 1_A$ and $A \in \mathcal{E}$, then the left- and right-hand sides of (11) are equal to the probabilities $P_{yx}(X_n \in A)$ and $P_x(X_n \in y^{-1}A)$, respectively, whence (11) follows in our case. Obviously, Eq. (11) remains valid if g is a Borel function taking finitely many values. If this condition is violated, then g can be uniformly approximated by functions satisfying this condition, and we again arrive at (11). \square

Consider the functions $h^B(x) = P_x(\Lambda_B^1)$ and $H^B(x) = P_x(\Lambda_B)$, where Λ_B^1 and Λ_B are the events that the trajectory of the random walk X hits a set $B \in \mathcal{E}$ at least once or infinitely many times, respectively.

Proposition 4. *One has*

$$(12) \quad h^{yB}(yx) = h^b(x), \quad H^{yB}(yx) = H^b(x)$$

for $x, y \in E$, where $yB = \{z = yx : x \in B\}$.

Proof. Most of the proof deals with the verification of the first relation in (12). First, let us verify that

$$(13) \quad h_n^{yB}(yx) = h_n^B(x), \quad n \geq 0,$$

for $x, y \in E$, where $h_0^B(x) = P_x(X_0 \in B)$ and

$$h_n^B(x) = P_x(X_0 \notin B, X_1 \notin B, \dots, X_{n-1} \notin B, X_n \in B), \quad n \geq 1.$$

Relation (13) is obvious for $n = 0$: both parts are simultaneously 1 or 0 depending on whether $x \in B$ or $x \notin B$, respectively. The case of $n \geq 1$ and $x \in B$ is equally easy, so we assume from now on that $x \notin B$. Assume that (13) has been proved for some $n \geq 0$. Then, by the Markov property,

$$(14) \quad h_{n+1}^{yB}(yx) = M_{yx}[1_\Lambda h_n^{yB}(X_1)]$$

with the factor 1_Λ that is the indicator function of the event $\Lambda = \{X_0 \notin B\}$. Since $x \notin B$, we have $P_{yx}(\Lambda) = P(yX_0 \notin yB) = 1$, and hence the right-hand side of (14) coincides with

$$M_{yx}[h_n^{yB}(X_1)] = M_x[h_n^{yB}(yX_1)] = M_x[h_n^B(X_1)] = h_{n+1}^B(x).$$

(Here the first equality follows from (11); the second, from (13) with the current value of n ; and the third, from (14) with $y = e$.) Thus, we have proved (13) with n replaced by $n + 1$, and so (13) holds for all $n \geq 0$.

Since $h^B(x) = \sum_{n \geq 0} h_n^B(x)$, we have simultaneously proved the first relation in (12), which, in conjunction with (11), implies the relations

$$(15) \quad M_{yx}[h^{yB}(X_n)] = M_x[h^{yB}(yX_n)] = M_x[h^B(X_n)].$$

Thus, the left- and right-hand sides of (15) coincide, and to justify the second relation in (12), it remains to note that $M_x[h^B(X_n)] \rightarrow H^B(x)$ as $n \rightarrow \infty$ ($x \in E$). \square

4. RANDOM WALKS R -RECURRENT IN THE SENSE OF TWEEDIE

The main goal of this section is to prove the second main result of this paper, Theorem 2. However, first we show how dramatically Theorem 1 and Corollary 1 are simplified if the random walk is spread out and R -recurrent in the sense of Tweedie, where, as before, R stands for the convergence parameter of X .

Proposition 5. *Under the above-mentioned conditions,*

- (i) *There exists a unique continuous exponential φ on E satisfying condition (1) with $r = R$.*
- (ii) *Every function g that is R -invariant for X or \hat{X} has the form indicated in the first or second assertion, respectively, of Corollary 1. Moreover, functions R -subinvariant for X or \hat{X} have the same form.*

Proof. One can readily establish that \hat{X} is spread out, irreducible, and hence π -irreducible. Let us prove that this random walk is R -recurrent in the sense of Tweedie. For a Borel function $f: E \rightarrow [0, \infty)$, set $G^R f = \sum_{n \geq 0} R^n P^n f$. We define $\hat{G}^R f$ with the use of the operator \hat{P} in a similar way. Assume that $\int f d\pi > 0$ and a function g has the same properties as f . Since $G^R f \equiv \infty$ owing to the R -recurrence of X , it follows that

$$(16) \quad \int f \hat{G}^R g d\pi = \int g G^R f d\pi = \infty,$$

and by setting $\nu = f\pi$, we obtain $\int \hat{G}^R g d\nu = 0$. By Proposition 3, we can subject the function f to the requirement that the measure ν thus introduced be small for \hat{X} . Now if $\hat{G}^R g \not\equiv \infty$, then the function $\hat{G}^R g$ is R -subinvariant for \hat{X} , and the last equality contradicts Proposition 5.1 in [7] provided that g is small for \hat{X} . Consequently, $\hat{G}^R g \equiv \infty$ for all g small for \hat{X} ; i.e., $\hat{R} \leq R$, where \hat{R} is the convergence parameter of the random walk \hat{X} . By interchanging X and \hat{X} , we obtain the inequality $R \leq \hat{R}$, and hence $R = \hat{R}$.

As a result, the random walk \hat{X} proves to be R -recurrent as well, which implies that Proposition 5 holds. (See [7, Theorem 5.3] as well as Theorem 1 and Corollary 1 in Sec. 2.) \square

Let us proceed to the main goal of this section. Starting from the law v of a Tweedie R -recurrent random walk X and the exponential φ mentioned in Proposition 5 (ii), consider a random walk \tilde{X} on E with the law $\tilde{v} = R\varphi v$ and the transition operator \hat{P} . (The measure \tilde{v} is a probability measure by virtue of the R -invariance of φ for X and condition (1) with $r = R$.) The convolution powers of the new law can readily be expressed via the same powers of v ,

$$(17) \quad \tilde{v}^n = R^n \varphi v^n, \quad n \geq 1.$$

For example, for each Borel function $f: E \rightarrow [0, \infty)$ we find that

$$\int f d(\tilde{v}^2) = R^2 \int f(xy) \varphi(x) \varphi(y) v(dx) v(dy) = R^2 \int f \varphi d(v^2),$$

which gives (17) for $n = 2$. (By induction, one can prove (17) with the use of similar computations for all $n \geq 1$.) In turn, (17) readily implies that

$$\tilde{P}^n f = R^n \frac{1}{\varphi} P^n(f\varphi), \quad n \geq 1,$$

which means that \tilde{P} can be obtained from P by a so-called similarity transformation [7]. (One also says that \tilde{X} is obtained from X by a passage to a φ -process [11].)

Theorem 2. *The random walk \tilde{X} is a Harris random walk [7], and hence the group E is recurrent.*

Proof. By [7, Proposition 5.3], \tilde{X} is π -irreducible and recurrent in the sense that $\tilde{H}^B > 0$ everywhere in E and $\tilde{H}^B = 1$ a.e. in E whenever $B \in \mathcal{E}_+$, where \tilde{H}^B is the counterpart of the function H^B (Proposition 5) for \tilde{X} . By [7, Proposition 3.13], the recurrence of \tilde{X} implies the existence of a Harris set $E_1 \in \mathcal{E}_+$, i.e., a set such that the restriction of \tilde{X} to E_1 is a Harris recurrent Markov chain.

Take an $x \in E$ and a $B \in \mathcal{E}_+$. If $z \in E_1$ and $y = zx^{-1}$, then, by Proposition 5,

$$\tilde{H}^B(x) = \tilde{H}^{yB}(yx) = \tilde{H}^{yB}(z),$$

and since $z \in E_1$ and $\pi(yB) > 0$, we have $\tilde{H}^{yB}(z) = 1$. In other words, $\tilde{H}^B(x) = 1$ for all $x \in E$, and hence the random walk \tilde{X} is a Harris random walk [7, Definition 3.5]. The proof of the theorem is complete. \square

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